Limit of a sequence

We want to write down a precise definition of what it means to say a sequence has a limit (and thus is convergent). As example, the sequence $\{1/n\} = \{1, 1/2, 1/3, 1/4, \ldots\}$ is convergent with the limit 0. We begin with an informal idea.

Rough idea: The number A is the limit of the sequence $\{a_n\}$ if, as n gets large, the elements a_n "settle down" so that A is the only reasonable value at "the end of the list".

To make this precise, we quantify what we mean by "large" values of the index n and we quantify what we mean by "settle down". We will use N to denote a specific index value that counts as "large". We will use ϵ to denote a measure of how close a_n is to A.

Precise idea: The number A is the limit of the sequence $\{a_n\}$ if for any positive measure ϵ, there is an index value N beyond which all elements a_n are within ϵ of A.

We can use inequalities to express this more compactly (and in a way that is easier to manipulate mathematically). Rather than writing "positive measure ϵ", we use $\epsilon > 0$. In place of writing "index value N beyond which", we use $n > N$. Finally, rather than writing "elements a_n are within ϵ of A, we use $|a_n - A| < \epsilon$.

Compact version: The number A is the limit of the sequence $\{a_n\}$ if for any $\epsilon > 0$, there is an index value N so that $n > N$ implies $|a_n - A| < \epsilon$.

Example: To prove that $A = 0$ is the limit of $\{a_n\} = \{1/n\}$, we start by considering a fixed value of $\epsilon > 0$. So ϵ is a given from which we need to construct (or show the existence of) an appropriate value of N. We need to find N to guarantee that $n > N$ implies $a_n - A| < \epsilon$. In this case, we need $|1/n - 0| < \epsilon$. This is equivalent to $n > 1/\epsilon$. So, any integer bigger than $1/\epsilon$ will work as a value of N. To be specific, we can choose N to be the smallest integer that is larger than $1/\epsilon$.

So, given any $\epsilon > 0$, we choose N to be the smallest integer larger than $1/\epsilon$ to have $N > 1/\epsilon$. If $n > N$, then $1/n < 1/N < \epsilon$. So $1/n < \epsilon$ which is equivalent to $|1/n - 0| < \epsilon$. Therefore 0 is the limit of $\{1/n\}$.